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A Diffusion Equation from the Relativistic Ornstein–Uhlenbeck Process

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We derive, in the "hydrodynamic" limit (large space and time scales), an evolution equation for the particle density in physical space from the (special) relativistic Ornstein–Uhlenbeck process introduced by Debbasch, Mallick, and Rivet. This equation turns out to be identical with the classical diffusion equation, without any relativistic correction. We prove that, in the "hydrodynamic" limit, this result is indeed compatible with special relativity.

KEY WORDS: Irreversibility; relativistic Ornstein–Uhlenbeck process; relativistic statistical physics; diffusion.

1. NOTATION

In this article, the velocity of light will be denoted by c and the signature of the space time metric will be chosen to be $(+, -, -, -)$.

The 4-position, 4-velocity and 4-momentum of the particle undergoing stochastic motion are denoted x^μ, u^μ and $p^\mu = mu^\mu$ respectively. $\mathbf{v}$ is the 3-velocity of the particle defined by the usual relation $\mathbf{v} = d\mathbf{x}/dt$. γ and m are respectively the Lorentz factor and the mass of the particle. U^μ designates the (local) 4-velocity of the surrounding fluid with which the particle interacts. As usual, Greek indices will run from 0 to 3 and Latin ones from 1 to 3.

Partial derivatives with respect to any variable a will be denoted ∂_a.

The modified Hankel function of order ν will be denoted $K_\nu(x)$ (see ref. 2 for definitions and properties).

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T will designate the absolute temperature, and k will stand for the Boltzmann constant.

In the asymptotic expansions, we have used the standard Landau's notations $\mathcal{O}(\varepsilon)$ and $o(\varepsilon)$ to designate quantities that are respectively of order ε and small compared to ε, when ε vanishes.

2. INTRODUCTION

In a recent article,[1] Debbasch, Mallick, and Rivet have introduced a new stochastic process which generalizes to the relativistic realm the usual Ornstein–Uhlenbeck process;[3,4] one of the most important features of this new process is its simplicity when compared to the continuous media theories which have been proposed until now to describe relativistic irreversible behavior;[5–7] indeed, one can view this stochastic process as a toy-model of relativistic irreversibility and it probably constitutes a very useful tool for investigating some possible limitations[6] of more usual approaches based on various relativistic generalizations of the Galilean Boltzmann and Navier–Stokes equations. At the very least, a systematic study of the process will certainly help put the foundations of relativistic statistical physics and hydrodynamics into a different perspective.

The principal aim of the present article is (i) to show that, in the "hydrodynamic" limit (large space and time scales), the particle density in physical space for the Relativistic Ornstein–Uhlenbeck Process (ROUP) obeys the same usual diffusion equation as in the Galilean case; (ii) to show that this equation, albeit parabolic, is indeed compatible with the principles of Einstein's relativity in the "hydrodynamic" limit.

In Section 3, we review rapidly some fundamentals about the ROUP already discussed at greater length in ref. 1. In particular, we introduce at this stage a Kramers' equation which fixes the time-evolution of the particle density in phase-space and also provides a hierarchy of evolution equations for various usual macroscopic fields such as the particle density and its associated current-density. Kramers' equation and this hierarchy play an even more pivotal role for the relativistic process than for its Galilean counterpart. Indeed, as we will explain in Section 3.2, they apparently provide the simplest way of obtaining informations on the long-time statistical behavior of the relativistic process. Technically, this is done by restricting the study to phase space distribution functions which vary "slowly" in space-time, and one can then search for them in the form of a Chapman–Enskog expansion around a so-called "local equilibrium" distribution. The hierarchy derived from Kramers' equation leads then to the usual diffusion equation for the particle-density in physical space. In addition, it provides a justification for the time- and space-scaling laws used in

the sequel. These matters are discussed in Section 4, including the precise definition of the hydrodynamic limit (Subsections 4.1 and 4.3). In order to get deeper insight on the problem, we were interested in finding the complete probability distribution, and not only the particle-density. The full Chapman–Enskog expansion of the solution of Kramers' equation can be found in Section 5 and is inspired by the work done by Wilemski,[8] Titulaer,[9,10] and Van Kampen[3] for the Galilean case. This expansion, which is naturally of interest *per se*, turns out to be necessary for discussing the consistency of the diffusion equation with the principles of Einstein's relativity. This is done in Section 6. As a conclusion, we mention some remaining problems left open for further study.

3. SOME FUNDAMENTALS ABOUT THE ROUP

Let us now review rapidly some general important points about the ROUP (Relativistic Ornstein–Uhlenbeck Process) which will be used extensively in the rest of this paper. A more substantial discussion of most of the issues raised in this section can be found in ref. 1.

3.1. Definition of the ROUP

The usual Ornstein–Uhlenbeck process can be viewed as a toy-model to describe the motion of a diffusing particle in interaction with a fluid. The interaction of this diffusing particle with the surrounding fluid is modeled by a deterministic friction-like force superimposed to a random Gaussian force (see ref. 1). The ROUP is an extension of the usual Galilean Ornstein–Uhlenbeck process to the relativistic realm.

The ROUP is defined by the following set of stochastic equations:

$$\begin{cases} \dfrac{d}{ds} x^\mu = u^\mu \\ \dfrac{d}{ds} p^\mu = -m\lambda^\mu_\nu (u^\nu - U^\nu) + m\lambda^\alpha_\beta u_\alpha (u^\beta - U^\beta)\, u^\mu + F^\mu \end{cases} \tag{1}$$

where s designates the proper time along the world line of the particle.

The first two terms on the right-hand side of (1) represent the deterministic "damping" part of the force experienced by the particle. It involves a second rank tensor λ which may generally depend on both u and U.

The stochastic contribution F is fully determined by the requirement that **F** be a centered Gaussian white-noise verifying:

$$\langle F^i(t_1)\, F_j(t_2)\rangle = -2D\delta(t_2 - t_1)\, \delta^i_j, \qquad D > 0$$

in the local rest-frame of the fluid, where **U** vanishes. The time component F^0 is then determined by the orthogonality condition of F and u.

For simplicity reasons, we will restrict the developments presented in this paper to cases where the space-time is Minkowskian and the surrounding fluid admits a global rest-frame which will be chosen as the reference frame where all calculations are done. Furthermore, we will assume that, in this frame, λ takes the simple spatially isotropic form:

$$\lambda_\nu^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

One then obtains from (1):

$$\begin{cases} \dfrac{d}{dt}\,\mathbf{x} = \dfrac{\mathbf{p}}{\gamma} \\ \dfrac{d}{dt}\,\mathbf{p} = -\alpha\gamma\mathbf{p} + \mathbf{F} \end{cases} \tag{2}$$

with $\gamma = \sqrt{1 + \mathbf{p}^2/m^2c^2}$. At this stage, α may be any scalar function of the velocity **u**. A physically sensible choice for this coefficient will be discussed below.

3.2. Kramers' Equation

Let $\Pi(t, \mathbf{x}, \mathbf{p})$ be the distribution function (in phase space) associated to the stochastic process (2). It has been shown in [1] that Π satisfies the following evolution equation, known as Kramers' equation:

$$\partial_t \Pi + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{p}}{\gamma m} \Pi \right) + \nabla_{\mathbf{p}} \cdot (-\alpha\gamma\mathbf{p}\Pi) = D\Delta_{\mathbf{p}}\Pi \tag{3}$$

This equation is naturally extremely difficult to solve for generic initial conditions; it can nevertheless furnish important informations about the diffusion process. A first simple implication of (3) has already been discussed in ref. 1: let $\Pi^{(eq)}$ be the usual relativistic Maxwell distribution with vanishing mean-velocity:

$$\Pi^{(eq)}(\mathbf{p}) = \frac{1}{4\pi\,(mc)^3} \frac{mc^2/kT}{K_2(mc^2/kT)} \exp\left(-\frac{mc^2}{kT}\,\gamma \right) \tag{4}$$

If one imposes the reasonable requirement that $\Pi^{(eq)}$ be a stationary solution of Kramers' equation, one obtains the physically correct expression for α in the form of a relativistic fluctuation-dissipation theorem:

$$\alpha = \frac{D}{mkT}\frac{1}{\gamma^2} \equiv \frac{\alpha_1}{\gamma^2} \tag{5}$$

where α_1 is a constitutive parameter of the model, independent of $\mathbf{p}$. When this choice for α is made, Kramers' equation reads:

$$\partial_t \Pi + \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{p}}{\gamma m}\Pi\right) - \nabla_{\mathbf{p}} \cdot \left(\alpha_1 \frac{\mathbf{p}}{\gamma}\Pi\right) - D\Delta_{\mathbf{p}}\Pi = 0 \tag{6}$$

In the Galilean regime ($\gamma = 1$), Eq. (2) is linear in $\mathbf{x}$ and $\mathbf{p}$. It can then be solved analytically,[4, 11] and the resulting solution contains all the information one needs on the process. In particular, one can show rigorously that $\mathbf{p}$ and $\mathbf{x}$ are Gaussian random variables whose mean values and variances can be evaluated exactly at any time. The Galilean fluctuation-dissipation theorem can notably be recovered in this way. In the relativistic regime, Eq. (2) is non-linear and cannot be solved exactly. This paradoxically makes of Kramers' equation the simplest tool we have at our disposal to study the stochastic process.

3.3. Dimensionless Kramers' Equation

To lighten further algebraic manipulations, we feel convenient to express the variables t, $\mathbf{x}$ and $\mathbf{p}$ in terms of natural units of time, position and momentum.

The natural time unit that comes directly out of (2) and (5) is α_1^{-1}. It represents the typical microscopic relaxation time of the stochastic process. We therefore choose as dimensionless time variable:

$$\underline{t} \equiv \alpha_1 t$$

Two natural velocity units could be chosen: the velocity of light c or the typical "thermal" velocity $(1/m)\sqrt{D/\alpha_1}$. The second choice has been retained since it is the only one that remains possible in the Galilean limit. Note also that the fluctuation-dissipation theorem (5) which, in some sense, defines the temperature in our model, leads to the usual expression $\sqrt{kT/m}$ for the velocity unit. The dimensionless velocity variable is consequently:

$$\underline{\mathbf{v}} \equiv m\sqrt{\frac{\alpha_1}{D}}\,\mathbf{v} \qquad \left(= \sqrt{\frac{m}{kT}}\,\mathbf{v}\right)$$

The natural space unit is simply the ratio of the velocity unit divided by the time unit: $(1/\alpha_1 m)\sqrt{D/\alpha_1}$ $(=(1/\alpha_1)\sqrt{kT/m})$. The dimensionless position variable is then:

$$\underline{\mathbf{x}} \equiv \alpha_1 m \sqrt{\frac{\alpha_1}{D}}\,\mathbf{x} \qquad \left(=\alpha_1 \sqrt{\frac{m}{kT}}\,\mathbf{x}\right)$$

The natural momentum unit is the velocity unit, multiplied by the mass m of the particle: $\sqrt{D/\alpha_1}$ $(=\sqrt{mkT})$. The dimensionless momentum variable is therefore:

$$\underline{\mathbf{p}} \equiv \sqrt{\frac{\alpha_1}{D}}\,\mathbf{p} \qquad \left(=\frac{\mathbf{p}}{\sqrt{mkT}}\right)$$

For further use, we denote by Q the ratio of the two aforementioned velocity units:

$$Q \equiv \frac{1}{mc}\sqrt{\frac{D}{\alpha_1}} \tag{7}$$

The dimensionless control parameter Q is of crucial importance, since it governs the proximity to the Galilean regime: high values of Q correspond to fully relativistic regimes, whereas the limit of vanishing Q is the Galilean regime. Moreover, the expression of Q in terms of the temperature T is, according to the fluctuation-dissipation theorem (5):

$$Q = \sqrt{\frac{kT}{mc^2}} \tag{8}$$

Under this form, it appears as the square root of the ratio of thermal energy to mass energy.

In term of these dimensionless variables, Kramers' equation reads:

$$\partial_{\underline{t}} \Pi + \nabla_{\underline{\mathbf{x}}} \cdot (\underline{\mathbf{p}}\gamma^{-1}\Pi) - \nabla_{\underline{\mathbf{p}}} \cdot (\underline{\mathbf{p}}\gamma^{-1}\Pi) = \Delta_{\underline{\mathbf{p}}}\Pi \tag{9}$$

with

$$\gamma = \sqrt{1 + Q^2\underline{\mathbf{p}}^2} \tag{10}$$

3.4. Hypotheses and Restrictions

Throughout this article, we will make the following assumptions:

1. When $|\mathbf{p}|$ tends to infinity, the probability distribution $\Pi(t, \mathbf{x}, \mathbf{p})$ and all its derivatives with respect to $\mathbf{p}$ vanish more rapidly than any power of $\mathbf{p}$, for all time and position:

$$\forall k, l, t, \mathbf{x}, \qquad \lim_{|\mathbf{p}| \to \infty} \mathbf{p}^k \partial_{p^l} \Pi(t, \mathbf{x}, \mathbf{p}) = 0$$

2. For technical simplicity reasons, we restrict our study to the one-dimensional case. We thus work with a one-dimensional probability distribution $\Pi(t, x, p)$ and with a one-dimensional restriction of (6):

$$\partial_t \Pi + \partial_x \left(\frac{p}{\gamma m} \Pi \right) - \partial_p \left(\alpha_1 \frac{p}{\gamma} \Pi \right) - D \partial_{pp} \Pi = 0 \tag{11}$$

or with its dimensionless version:

$$\partial_{\underline{t}} \Pi + \partial_{\underline{x}} (p \gamma^{-1} \Pi) - \partial_{\underline{p}} (\underline{p} \gamma^{-1} \Pi) - \partial_{\underline{p}\underline{p}} \Pi = 0 \tag{12}$$

A complete three-dimensional calculation would certainly not reveal new physics, but would obviously generate more cumbersome algebra.

3. The typical linear size $\mathscr{L}$ of the accessible region in physical space will be assumed to be finite although very large compared to any physically relevant length scale of the problem. The volume $\mathscr{V} \simeq \mathscr{L}^3$ of this region is thus finite; this gives a well-defined meaning to the notion of uniform particle-density in physical space.

The main point of this article is to derive from (11), under the above listed hypotheses, an equation of evolution for the particle density in physical space, in the so-called "hydrodynamic" limit. This goal will be achieved by a Chapman–Enskog expansion around a so-called "local equilibrium" distribution. The result will be presented and discussed in the next three sections.

4. THE MOMENTUM HIERARCHY

The complete statistical information contained in the probability distribution function $\Pi(\underline{t}, \underline{x}, \underline{p})$ can be translated via (12) into a hierarchy of evolution equations for the momenta of Π. To do this, one multiplies (12) by $\underline{p}^k$ (k is a positive integer), and take the average over the possible values of $\underline{p}$. The terms containing derivatives of Π with respect to $\underline{p}$ can be integrated by parts, and Hypothesis number 1 guarantees that the integrated

parts vanish. Finally, one obtains the following hierarchy of evolution equations for the momenta of Π:

$$\partial_t(n\langle \underline{p}^k\rangle) + \partial_{\underline{x}}(n\langle \underline{p}^{k+1}\gamma^{-1}\rangle) + kn\langle \underline{p}^k\gamma^{-1}\rangle - k(k-1)\, n\langle \underline{p}^{k-2}\rangle = 0, \qquad \text{for} \quad k \geqslant 0 \tag{13}$$

where the symbol $\langle\ \rangle$ designates the average over $\underline{p}$:

$$n\langle f\rangle \equiv \int_{-\infty}^{+\infty} f(\underline{p})\, \Pi(\underline{t}, \underline{x}, \underline{p})\, d\underline{p}$$

We can also write the above hierarchy under the following form:

$$\partial_t(n\langle \underline{p}^k\rangle) + \partial_{\underline{x}}(n\langle \underline{v}\underline{p}^k\rangle) + kn\langle \underline{v}\underline{p}^{k-1}\rangle - k(k-1)\, n\langle \underline{p}^{k-2}\rangle = 0, \qquad \text{for} \quad k \geqslant 0 \tag{14}$$

where $\underline{v} \equiv \underline{p}/\gamma$ is the dimensionless velocity of the particle. Note that (14), contrary to (13) is valid in both Galilean and relativistic cases.

As it is usually done in the Galilean case, we use a scale separation argument ("hydrodynamic" limit) and a Chapman–Enskog expansion[10, 12] to close the hierarchy (14).

4.1. The "Hydrodynamic" Limit

We need to face a situation frequently encountered in Statistical Mechanics: we have a system described by a "microscopic" evolution law (2) and we would like to derive from it a "macroscopic" description of its collective behavior. To do this, we make the standard "hydrodynamic" limit which consists in restricting our study to situations where the probability distribution $\Pi(\underline{t}, \underline{x}, \underline{p})$ is nearly homogeneous (space-independent) for any $(\underline{t}, \underline{p})$, and close to an equilibrium distribution for any $(\underline{t}, \underline{x})$. It will be shown in Section 6 that, somewhat expectedly, the long-time statistical behavior of the process can be captured by this approach.

More precisely, let us define the space-scale separation parameter ε by:

$$\varepsilon \equiv \max_{(\underline{t}, \underline{x}, \underline{p})} \left(\left| \frac{\partial_{\underline{x}} \Pi}{\Pi} \right| \right) \tag{15}$$

The assumption that $\varepsilon \ll 1$ gives a quantitative meaning to the phrase "nearly homogeneous for any $(\underline{t}, \underline{p})$".

A similar scale separation parameter for the time-variations will be introduced later and linked very simply to ε.

4.2. The Chapman–Enskog Expansion

When Π is nearly homogeneous, so is the particle-density $n(\underline{t}, \underline{x})$. The latter is hereby close to the uniform particle-density $n_0 \equiv \mathscr{L}^{-1}$, where $\mathscr{L}$ is the typical linear dimension of the (one-dimensional) accessible domain in physical space (see Hypothesis number 3). Let us now introduce the "global equilibrium distribution" $\Pi^{(eq)}(\underline{p})$ defined by:

$$\Pi^{(eq)}(\underline{p}) = \frac{1}{\mathscr{L}} \frac{Q}{2K_1(Q^{-2})} \exp(-\gamma/Q^2) \tag{16}$$

where $\gamma = \sqrt{1 + Q^2\underline{p}^2}$. This distribution is identical to (4), but with a one dimensional normalization in $\underline{p}$ and $\underline{x}$:

$$\int_{-\mathscr{L}/2}^{\mathscr{L}/2} \int_{-\infty}^{\infty} \Pi^{(eq)}(\underline{p})\, d\underline{p}\, d\underline{x} = 1 \tag{17}$$

$\Pi^{(eq)}$ is an homogeneous and stationary solution of Kramers' equation (12). The name "global equilibrium" has been chosen for consistency with the usual vocabulary of Statistical Mechanics.

It is now natural to consider that the lowest order approximation $\Pi^{(0)}$ for $\Pi(\underline{t}, \underline{x}, \underline{p})$ will be of the form (16), but with the uniform particle-density $1/\mathscr{L}$ replaced by the real particle-density $n(\underline{t}, \underline{x})$:

$$\Pi^{(0)}(\underline{t}, \underline{x}, \underline{p}) = n(\underline{t}, \underline{x}) \frac{Q}{2K_1(Q^{-2})} \exp(-\gamma/Q^2) \tag{18}$$

This distribution, which is *not* an exact solution of Kramers' equation, is analogous to what is usually called a "local equilibrium" in Statistical Mechanics. This terminology will be kept here for historical consistency, although it is slightly misleading: indeed, $\Pi^{(0)}$ is *not* an equilibrium, since it is not a solution of Kramers' equation. It is also worth noting that since $\Pi^{(0)}$ has the same analytical dependence on $\underline{p}$ as $\Pi^{(eq)}$, we have:

$$\int_{-\infty}^{\infty} \Pi^{(0)}\, d\underline{p} = n(\underline{t}, \underline{x})$$

Following the idea of the usual Chapman–Enskog expansion, we seek for solutions of Kramers' equation (12) under the form:

$$\Pi(\underline{t}, \underline{x}, \underline{p}) = \sum_{I=0}^{N} \varepsilon^I \Pi^{(I)}(\underline{t}, \underline{x}, \underline{p}) + \mathcal{O}(\varepsilon^{N+1}) \tag{19}$$

where the zeroth order term $\Pi^{(0)}$ is precisely the local equilibrium distribution defined above. By construction, the integral of Π over $\underline{p}$ must be $n(\underline{t}, \underline{x})$. Since the integral of $\Pi^{(0)}$ over $\underline{p}$ must also give $n(\underline{t}, \underline{x})$, we obtain immediately that the integral over $\underline{p}$ of all subsequent terms $\Pi^{(I)}$ in the expansion (19) must vanish.

The average of any function $f(\underline{p})$ can also be expanded in powers of ε:

$$n\langle f\rangle = \sum_{I=0}^{N} \varepsilon^I n\langle f\rangle^{(I)} + \mathcal{O}(\varepsilon^{N+1}) \tag{20}$$

where each term is simply:

$$n\langle f\rangle^{(I)} = \int_{-\infty}^{\infty} f(\underline{p})\, \Pi^{(I)}(\underline{t}, \underline{x}, \underline{p})\, d\underline{p}$$

For the particular case of the momentum of order 0, namely n, we find that the expansion (20) reduces to its zeroth order term, because, for any $I > 0$, the integral of $\Pi^{(I)}$ over $\underline{p}$ vanishes.

For odd functions of $\underline{p}$, the first term of the expansion (20) is zero since the local equilibrium distribution (18) is even in $\underline{p}$. Thus, the average of all odd functions of $\underline{p}$ is $\mathcal{O}(\varepsilon)$ or smaller. This remark will ease the obtaining of the proper scaling for the time-dependence of Π and n.

4.3. The Time Dependence of n

If the particle-density n satisfies the hypothesis of the "hydrodynamic" limit, namely that its space-variations are slow, then its time-evolution will be slow too, with a time-scale separation η that we now prove to be $\mathcal{O}(\varepsilon^2)$. In the spirit of (15), we define the time-scale separation η by:

$$\eta \equiv \max_{(\underline{t}, \underline{x}, \underline{p})} \left(\left| \frac{\partial_{\underline{t}} \Pi}{\Pi} \right| \right) \tag{21}$$

Let us consider the levels $k = 0$ and $k = 1$ in the hierarchy (14):

$$\partial_{\underline{t}} n + \partial_{\underline{x}} n\langle \underline{v}\rangle = 0 \tag{22}$$

$$\partial_{\underline{t}} n\langle \underline{p}\rangle + \partial_{\underline{x}} n\langle \underline{v}\underline{p}\rangle + n\langle \underline{v}\rangle = 0 \tag{23}$$

We define an auxiliary (small) parameter ε' by $n\langle \underline{v} \rangle = \mathcal{O}(\varepsilon')$. Equation (23) gives immediately $\eta = \varepsilon\varepsilon'$ and, since $n\langle \underline{v}\underline{p} \rangle = \mathcal{O}(\varepsilon^0)$, Eq. (23) then implies $\varepsilon = \varepsilon'$. This yields:

$$\eta = \mathcal{O}(\varepsilon^2)$$

which is the standard diffusive time-scaling.

4.4. The Equation of Diffusion for *n*

To derive an equation of evolution for n, we first collect all terms of order ε from the level $k = 1$ equation in the hierarchy (14):

$$\partial_{\underline{x}}(n\langle \underline{v}\underline{p} \rangle^{(0)}) + n\langle \underline{v} \rangle^{(1)} = 0$$

The quantity $n\langle \underline{v}\underline{p} \rangle^{(0)}$ only involves $\Pi^{(0)}$ whose analytic form is (18). A simple integration by parts shows that $n\langle \underline{v}\underline{p} \rangle^{(0)} = n$, provided that Hypothesis number 1 holds. This leads immediately to an expression for $n\langle \underline{v} \rangle^{(1)}$:

$$n\langle \underline{v} \rangle^{(1)} = -\partial_{\underline{x}} n \tag{24}$$

Let us now consider the level $k = 0$ equation in the hierarchy (14), from which we collect separately for all $J > 2$ all terms of order ε^J. This leads to:

$$\partial_{\underline{t}} n + \partial_{\underline{x}} n\langle \underline{v} \rangle^{(1)} = 0 \quad \text{and} \quad \partial_{\underline{x}} n\langle \underline{v} \rangle^{(I)} = 0, \qquad \text{for all} \quad I > 1$$

Combining the above result with (24), we obtain:

$$\partial_{\underline{t}} n - \partial_{\underline{x}\underline{x}} n = 0 \tag{25}$$

which is valid at any order. We now use the space- and time-units described in Section 3.3 to restore the original variables t and x. We obtain the standard diffusion equation:

$$\partial_t n - \chi \partial_{xx} n = 0 \tag{26}$$

where the diffusion coefficient χ is $D/m^2\alpha_1^2$ or, using the fluctuation-dissipation theorem: $\chi = kT/m\alpha_1$.

At this point, we feel necessary to emphasize the fact that the arguments presented in this section are systematic but not rigorous from

the mathematical point of view, since the convergence of the Chapman–Enskog expansion (19) has not been proven.

5. PERTURBATIVE RESOLUITON OF KRAMERS' EQUATION

In Section 4, we have worked in the so-called "hydrodynamic" limit and used a Chapman–Enskog expansion to obtain results on the marginal n of Π, and not on Π itself. We now address the problem of solving directly Kramers' equation (12), by the same Chapman–Enskog expansion in powers of the scale separation parameter ε. This extension to Special Relativity of the work presented in refs. 3 and 8–10 will be most useful in discussing Eq. (26) in the next section.

We start from (12) written under the form:

$$\partial_{\underline{p}}(\underline{v}\Pi + \partial_{\underline{p}}\Pi) = \partial_{\underline{t}}\Pi + \underline{v}\partial_{\underline{x}}\Pi \tag{27}$$

where $\underline{v} = \underline{p}/\gamma$. As in Section 4, we use the "hydrodynamic" limit based upon the scale separation parameter ε. We thus introduce new rescaled space- and time-variables $\underline{x}' = \varepsilon\underline{x}$ and $\underline{t}' = \varepsilon^2\underline{t}$, to rewrite (27) as:

$$\partial_{\underline{p}}(\underline{v}\Pi + \partial_{\underline{p}}\Pi) = \varepsilon^2\partial_{\underline{t}'}\Pi + \underline{v}\varepsilon\partial_{\underline{x}'}\Pi \tag{28}$$

We will again make use of the Chapman–Enskog expansion (19) for Π:

$$\Pi(\underline{t}, \underline{x}, \underline{p}) = \sum_{I=0}^{N} \varepsilon^I \Pi^{(I)}(\underline{t}, \underline{x}, \underline{p}) + \mathcal{O}(\varepsilon^{N+1})$$

where $\Pi^{(0)}$ is a local equilibrium distribution function (see Section 4.2). We now examine the orders ε^0, ε^1, ε^2, to guess a systematic form for $\Pi^{(k)}$. This guessed systematic form will then be proven by recursion to be valid at any order.

5.1. Order ϵ^0

We use the expanded expression (19) into Kramers' equation (28) and collect all terms of order ε^0. This leads to:

$$\partial_{\underline{p}}(\underline{v}\Pi^{(0)} + \partial_{\underline{p}}\Pi^{(0)}) = 0 \tag{29}$$

The only solutions of (29) compatible with Hypothesis number 14 are of the form:

$$\Pi^{(0)} = A^{(0)} \exp(-\gamma/Q^2) \tag{30}$$

where $A^{(0)}$ is a function of $\underline{t}$ and $\underline{x}$ which we identify to $n(\underline{x}, \underline{t})\, Q/2K_1(Q^{-2})$ for consistency with (18) (see discussion in Section 4.2).

5.2. Order ϵ^1

We collect all terms of order ε^1 in (28) and give to $\Pi^{(0)}$ the form (30) derived above. We obtain:

$$\partial_{\underline{p}}(\underline{v}\Pi^{(1)} + \partial_{\underline{p}}\Pi^{(1)}) = \underline{v}\partial_{\underline{x}'}A^{(0)} \exp(-\gamma/Q^2) \tag{31}$$

The only solutions of (31) compatible with Hypothesis number 1 are of the form:

$$\Pi^{(1)} = (A^{(1)} - \underline{p}\partial_{\underline{x}'}A^{(0)}) \exp(-\gamma/Q^2) \tag{32}$$

where $A^{(1)}$ is a function of $\underline{t}'$ and $\underline{x}'$. Up to this order, no solubility condition has to be imposed to get solutions verifying Hypothesis number 1. This is no more the case at next order.

5.3. Order ϵ^2

We collect all terms of order ε^2 in (28) and give to $\Pi^{(0)}$ and $\Pi^{(1)}$ the forms (30) and (32) derived above. We obtain:

$$\partial_{\underline{p}}(\underline{v}\Pi^{(1)} + \partial_{\underline{p}}\Pi^{(1)}) = (\partial_{\underline{t}'}A^{(0)} + \underline{v}\partial_{\underline{x}'}A^{(1)} - \underline{v}p\partial_{\underline{x}'\underline{x}'}A^{(0)}) \exp(-\gamma/Q^2) \tag{33}$$

To satisfy Hypothesis number 1, the integral over $\underline{p}$ of the right-hand side must vanish. Indeed, let us take the primitive of both sides of (33). The primitive of the left-hand side clearly vanishes when $\underline{p}$ tends to infinity if Hypothesis number 1 is verified. Thus, the primitive of the right-hand side must also vanish when p tends to infinity. Hence, the integral over $\underline{p}$ of the right-hand side must be zero. To compute this integral, we use a relation whose proof is straightforward:

$$\frac{d}{d\underline{p}}(\underline{p}^k \exp(-\gamma/Q^2)) = (k\underline{p}^{k-1} - \underline{v}\underline{p}^k) \exp(-\gamma/Q^2), \qquad \text{for} \quad k \geqslant 0 \tag{34}$$

From (34), we find that a necessary (and sufficient) condition for the integral over $\underline{p}$ of the right-hand side of (33) to vanish is:

$$\partial_{\underline{t}'}A^{(0)} - \partial_{\underline{x}'\underline{x}'}A^{(0)} = 0 \tag{35}$$

This solubility condition gives back the diffusion equation (26) for the particle-density n, since $A^{(0)}$ is simply proportional to n.

Assuming this condition to be fulfilled, the only solutions of (33) compatible with Hypothesis number 1 are of the form:

$$\Pi^{(2)} = \left(A^{(2)} - \underline{p}\partial_{\underline{x}'} A^{(1)} + \frac{\underline{p}^2}{2} \partial_{\underline{x}'\underline{x}'} A^{(0)} \right) \exp(-\gamma/Q^2) \tag{36}$$

As we shall see, the same structure extends to any subsequent order.

5.4. Order ϵ^k

We now prove by recursion that, at any order $k \geqslant 2$:

- the solubility condition is:

$$\partial_{\underline{t}'} A^{(k-2)} - \partial_{\underline{x}'\underline{x}'} A^{(k-2)} = 0 \tag{37}$$

- the solution is[3]:

$$\Pi^{(k)} = \exp(-\gamma/Q^2) \sum_{m=0}^{k} \frac{(-\underline{p})^m}{m!} \partial_{\underline{x}'^m} A^{(k-m)} \tag{38}$$

We assume that the above statements are true for orders 0 to k, and study the case $k+1$. We thus gather the terms of order ε^{k+1} in Kramers' equation (28), to get:

$$\partial_{\underline{p}}(\underline{v}\Pi^{(k+1)} + \partial_{\underline{p}}\Pi^{(k+1)}) = \partial_{\underline{t}'}\Pi^{(k-1)} + \underline{v}\partial_{\underline{x}'}\Pi^{(k)}$$

We express the right-hand side of this equation using (38) at orders k and $k-1$. We obtain:

$$\partial_{\underline{p}}(\underline{v}\Pi^{(k+1)} + \partial_{\underline{p}}\Pi^{(k+1)})$$
$$= \left[\sum_{m=0}^{k-1} \frac{(-\underline{p})^m}{m!} \partial_{\underline{t}'} \partial_{\underline{x}'^m} A^{(k-1-m)} + \sum_{m=0}^{k} \underline{v} \frac{(-\underline{p})^m}{m!} \partial_{\underline{x}'^{m+1}} A^{(k-m)} \right] \exp(-\gamma/Q^2)$$

We then use the solubility conditions (37) for orders 0 to $k-2$ and get, after some simple index manipulations:

[3] Note that here, m is an integer and not the mass of the diffusing particle.

$$\partial_{\underline{p}}(\underline{v}\Pi^{(k+1)} + \partial_{\underline{p}}\Pi^{(k+1)})$$
$$= \Bigg[\partial_{\underline{t}'} A^{(k-1)} + \underline{v}\partial_{\underline{x}'} A^{(k)} - \underline{v}\underline{p}\partial_{\underline{x}'\underline{x}'} A^{(k-1)}$$
$$- \sum_{m=2}^{k} \frac{(-1)^m}{m!} (m\underline{p}^{m-1} - \underline{v}\underline{p}^m)\, \partial_{\underline{x}'^{m+1}} A^{(k-m)} \Bigg] \exp(-\gamma/Q^2)$$

Exactly as for order ε^2, the solubility condition is that the integral over p of the right-hand side of this equation must be zero. The identity (34) can be used to evaluate this integral and to prove that it vanishes if and only if:

$$\partial_{\underline{t}'} A^{(k-1)} - \partial_{\underline{x}'\underline{x}'} A^{(k-1)} = 0 \tag{39}$$

If this solubility condition for the order ε^{k+1} is assumed, then the equation for $\Pi^{(k+1)}$ reduces to:

$$\partial_{\underline{p}}(\underline{v}\Pi^{(k+1)} + \partial_{\underline{p}}\Pi^{(k+1)})$$
$$= \Bigg[\underline{v}\partial_{\underline{x}'} A^{(k)} - \sum_{m=1}^{k} \frac{(-1)^m}{m!} (m\underline{p}^{m-1} - \underline{v}\underline{p}^m)\, \partial_{\underline{x}'^{m+1}} A^{(k-m)} \Bigg] \exp(-\gamma/Q^2)$$

Using again the identity (34), we find that the only solution of this equation, that satisfy Hypothesis number 1 takes the form:

$$\Pi^{(k+1)} = \exp(-\gamma/Q^2) \sum_{m=0}^{k+1} \frac{(-\underline{p})^m}{m!}\, \partial_{\underline{x}'^m} A^{(k+1-m)} \tag{40}$$

which is nothing else but (38) with k changed into $k+1$. This terminates the recursive proof that, for any order I, the probability distribution Π solution of Kramers' equation is of the form[4]:

$$\Pi(\underline{t}, \underline{x}, \underline{p}) = \sum_{k=0}^{I} \varepsilon^k \exp(-\gamma/Q^2) \sum_{m=0}^{k} \frac{(-\underline{p})^m}{m!} \varepsilon^{-m} \partial_{\underline{x}^m} A^{(k-m)} + \mathcal{O}(\varepsilon^{I+1}) \tag{41}$$

with

$$\partial_{\underline{t}} A^{(k)} - \partial_{\underline{x}\underline{x}} A^{(k)} = 0, \qquad \text{for} \quad 0 \leqslant k \leqslant I-2 \tag{42}$$

[4] Note that we have re-expressed below the rescaled variables $\underline{t}'$ and $\underline{x}'$ in terms of the original dimensionless time- and space-variables $\underline{t}$ and $\underline{x}$.

and

$$A^{(0)} = n(\underline{t}, \underline{x}) \tag{43}$$

The additional condition that $\int \Pi \, d\underline{p} = n$ gives a simple way to completely determine the $A^{(k)}$ in terms of $A^{(0)}$. In particular, this condition implies that $A^{(1)}$ identically vanishes. This remark will be used in the next section.

If we now assume that the series (41) for Π converges, a fact which has not been proven, it is then easy to see that the solubility conditions (42) impose that:

$$\partial_{\underline{t}} \Pi - \partial_{\underline{x}\underline{x}} \Pi = 0 \tag{44}$$

The same result holds if we use for Π the expression (41) for any finite value of I, i.e., if we conserve only a finite number of terms in the aforementioned series. We now restore the physical variables t and x instead of $\underline{t}$ and $\underline{x}$ to obtain:

$$\partial_t \Pi - \chi \partial_{xx} \Pi = 0 \tag{45}$$

where the diffusion coefficient χ is $D/m^2\alpha_1^2$ or, using the fluctuation-dissipation theorem: $\chi = kT/m\alpha_1$.

6. CONSISTENCY WITH THE PRINCIPLES OF EINSTEIN'S RELATIVITY

We would like now to discuss thoroughly the somewhat startlingly simple result(s) obtained in the preceding section.

The first question one naturally asks is how the relativistic stochastic system (2) can lead to the parabolic equations (26) and (45), which can, loosely speaking, "propagate matter at unbounded velocities." To investigate this problem, let us focus our attention on either one of the two equations concerned, say (26), and start the analysis by following the temporal evolution of a statistical ensemble with initial spatial density $\nu_0(x) = \delta(x)$. The corresponding solution of (26) is then $G(t, x)$, the Green function of the diffusion equation, given by the familiar expression:

$$G(t, x) = \frac{1}{\sqrt{4\pi\chi t}} \exp\left(-\frac{x^2}{4\chi t}\right) \tag{46}$$

Let us stress at this point that, if (26) is valid, any ensemble with initial spatial density ν_0 will produce, at time t, an ensemble with spatial density $G(t, x)$, whatever the initial momentum distribution is. An apparent contradiction with Special Relativity seems to arise because, for any $t > 0$, $G(t, x)$

does not vanish outside the light-cone defined by $|x| < ct$. To understand this seemingly troublesome situation, let us take a closer look at how the "hydrodynamic" scalings involved in deriving (26) from Kramers' equation work on $G(t, x)$. The spatial scaling, when applied to G, gives simply:

$$\left|\frac{x}{t}\right| = \sqrt{\frac{kT}{m}}\,\mathcal{O}(\varepsilon) \tag{47}$$

To extract from this relation some more interesting information, let us consider again the Chapman–Enskog expansion (19) of Π. For finite value of ε, a necessary (but not sufficient) condition for the whole expansion procedure to make sense is that the order of magnitude of each term in (19) is much smaller than the order of magnitude of the preceding one. To give a precise meaning to the expression "order of magnitude", one has to choose a norm on a suitable space which contains the functions under consideration. Because of the general form of $\Pi^{(k)}$, we retain the following definition for the norm $\mathcal{N}(f)$ of any function $f(\underline{t}, \underline{x}, \underline{p})$ to be considered here:

$$\mathcal{N}(f) = \max_{(\underline{t}, \underline{x})} \int_{-\infty}^{\infty} |f|\, d\underline{p} \tag{48}$$

If one then writes that $\mathcal{N}(\varepsilon\Pi^{(1)}) \ll \mathcal{N}(\Pi^{(0)})$, one obtains from (30) and (32)[5]:

$$\varepsilon \int_{-\infty}^{\infty} |\underline{p}| \exp(-\gamma/Q^2)\, d\underline{p} \ll \int_{-\infty}^{\infty} \exp(-\gamma/Q^2)\, d\underline{p}$$

Expliciting the integrals, this relation gives:

$$\varepsilon Q \ll h(Q) \tag{49}$$

where $h(Q)$ is given by:

$$h(Q) = \frac{\exp(Q^{-2})\, K_1(Q^{-2})}{(1+Q^2)}$$

This positive, continuous function is defined for all non-vanishing values of Q. Moreover, analytical equivalents of $K_1(z)$ for $z \gg 1$ and $z \ll 1$ lead to the conclusion that $h(Q)$ tends linearly to zero when Q tends to zero and tends to 1 when Q tends to infinity. h is therefore bounded by a positive constant h_0. A numerical study shows that h is actually monotonously

[5] Note that typically, $\partial_{\underline{x}} A^{(0)}$ and $A^{(0)}$ are both of the same order of magnitude, which has to be considered $\mathcal{O}(\varepsilon^0)$.

increasing. We can therefore choose $h_0 = 1$. One deduces then from (49):

$$\varepsilon Q \ll 1$$

which in turn implies via (47):

$$\left|\frac{x}{t}\right| \ll c \tag{50}$$

$|x/ct|$ has therefore to be a vanishingly small quantity for $G(t, x)$ to represent adequately the marginal of the "true" phase-space distribution at (t, x) obtained by solving Kramers' equation (12). This naturally ensures the consistency of (26) with Special Relativity. The same reasoning applies immediately to (45) and also proves its consistency with Special Relativity. It is interesting to remark that in the Galilean regime ($Q \ll 1$), (49) degenerates into $\varepsilon \ll 1$ which is identical with the original assumption found in Section 4.1.

Let us also note that the scaling on the time evolution of Π, when applied to G, gives simply:

$$-\frac{1}{2t} + \frac{x^2}{4\chi t^2} = \mathcal{O}(\eta)$$

which delivers, with the help of (47):

$$\frac{1}{t} = \alpha_1 \left(1 + \frac{\chi}{2}\right) \mathcal{O}(\eta)$$

This explicitly proves that the scaling under consideration is essentially a "long-time" scaling.

The reason why the preceding results are so important is that the Green function G can be used to generate all (sufficiently regular) solutions of (26) More precisely, if $v(x)$ is any regular distribution, the solution of (26) which satisfies $n(t = 0, x) = v(x)$ is given by the convolution:

$$n(t, x) = \int_{-\infty}^{\infty} v(y)\, G(t, x - y)\, dy \tag{51}$$

To explore the consequences of (51), let us suppose the support of v to be compact and centered on $x = 0$, with typical extension L. The "hydrodynamic" scalings then deliver the conditions:

$$\frac{1}{t} = \alpha \left(1 + \frac{\chi}{2}\right) \mathcal{O}(\eta) \tag{52}$$

and, for all y in $[-L/2, +L/2]$,

$$\left|\frac{x-y}{t}\right| = \sqrt{\frac{kT}{m}}\,\mathcal{O}(\varepsilon) \tag{53}$$

For finite L, the preceding relation states as before that, when considered in the domain where the scaling (15)/(21) and the whole expansion procedure apply, no solution of (26) displays any violation of Special Relativity. If L is not finite, i.e., if the support of the initial spatial density is not bounded, (53) proves that there is no point in space-time where an "hydrodynamic" description of the stochastic process is adequate. It is important to stress that this remark also applies in the Galilean limit.

To sum up the preceding discussion: Equation (45) and (26) are naturally parabolic and, if considered by themselves, violate Einstein's relativity in the sense that, loosely speaking, they can propagate spatial inhomogeneities at infinite velocity. However, if one considers them as consequences of the (exact) Kramers' equation associated to the ROUP, (45) and (26) are to be accompanied by restriction on the domain of space-time where they can be used. These restrictions automatically prevent any contradiction between the solutions of (45) and (26) and Special Relativity. In particular, the Green function of (26) can be considered to describe adequately the diffusion in physical space of an ensemble of particles all located initially at the origin of coordinates only for vanishingly small values of x/ct. More precisely, when considered as a consequence of Kramers' equation, the diffusion equation can only be used in region of space-time where it "propagates" inhomogeneities at vanishingly low velocities compared to both c and the mean thermal velocity. This important conclusion is valid in both Galilean and relativistic regimes.

It is worth noting that, as n, $\Pi(t, x, p)$ itself (without any integration over p) satisfies the diffusion equation (45) at any order in ε. This property, which is verified in both Galilean and special relativistic regimes, do not seem to have found any explanation in the existing literature.

7. CONCLUSION

We have investigated more closely the ROUP in the rest-frame of the fluid in which the particle diffuses. Under the physical hypothesis that all momenta of the distribution function in phase space are "slowly-varying" fields (in space-time), we have developed a systematic expansion somewhat similar to the one already proposed by Wilemski,[8] Titulaer,[9,10] and Van Kampen;[3] although both expansions deliver the same results in the Galilean regime, the one introduced in this paper is also valid in the

relativistic one and rests on slightly different conceptual foundations. We have then proved that the relativistic diffusion in physical space can be adequately described, at any order of the expansion, by the same parabolic diffusion equation as the usual Galilean one. This striking result is in no contradiction with Special Relativity; indeed, the procedure used to derive the diffusion equation from the exact evolution equation for the density in phase space makes clear that the diffusion equation is only valid in regions of space-time where it diffuses density inhomogeneities at vanishingly low velocities. This important conclusion also applies to the usual (Galilean) Ornstein–Uhlenbeck process.

In the context considered in this article, the various extra-terms proposed by several authors for modeling relativistic heat transfer and diffusion[7, 13, 14] appear therefore to be unnecessary. In other words, to incorporate relativistic effects does not change the equation which governs the evolution of slowly varying inhomogeneities near an equilibrium situation. It is most probable that the same conclusion also holds if the diffusion equation is derived from the usual Galilean and Special Relativistic kinetic theory.

If the system is too far from an equilibrium situation, the "slowly-varying" fields-expansion considered in this article can surely be of no use. Titulaer[9] has developed, in the Galilean case, other expansions which permit a satisfactory treatment of the problem. These expansions have not been extended to the relativistic case in this paper but work in this direction is currently under way. It is to be noted that, in the Galilean case, these other expansions do predict the appearance of extra-terms in the diffusion equation, all of them built out of higher momenta of the distribution function in phase space. It is more than likely that this will also be the case in the relativistic regime. The results of these expansions, for both regimes, should be systematically compared to the so-called "Extended Thermodynamics" theories,[6, 7] which, based on kinetic theory, introduce axiomatically in continuum mechanics fields which are not taken as basic quantities by the more usual theories and which essentially reflect higher momenta of the distribution function. What this article has proved is that these fields, even in the relativistic case, have not to be taken into account if all momenta of the distribution function vary sufficiently slowly in space-time.

Finally, the extension of the results presented here to other reference frames and to the general relativistic framework is also of utmost importance.

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